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The Hagedorn transition, Deconfinement and $\mathcal{N} = 4$ SYM Theory

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Abstract

$\mathcal{N} = 4$ Super Yang-Mills theory supplies us with a non-Abelian 4D gauge theory with a meaningful perturbation expansion, both in the UV and in the IR. We calculate the free energy on a 3-sphere and observe a deconfinement transition for large N at zero coupling. The same thermodynamic behaviour is found for a wide class of toy models, possibly also including the case of non-zero coupling. Below the transition we also find Hagedorn behaviour, which is identified with fluctuations signalling the approach to the deconfined phase. The Hagedorn and the deconfinement temperatures are identical. Application of the AdS/CFT correspondence gives a connection between string Hagedorn behaviour and black hole formation.

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1 Introduction

In large N discussions of Yang-Mills theory the thermal deconfinement transition is accompanied by a jump in the free energy from order N^0 to order N^2 . The confined phase consists of hadrons and glueballs, which should have spectra with narrow resonances. Models for such spectra like statistical bootstrap models or string theory, often have densities of states rising exponentially with energy, i.e. Hagedorn behaviour [1]. Hagedorn spectra themselves give rise to singular thermodynamics at a characteristic temperature because Boltzmann suppression of high energy states can be compensated by the sheer numbers of states. It would be natural if this Hagedorn temperature and the deconfinement temperature were the same. Thorn [2] has discussed possible alternatives but it seems that the question for the generic 4-dimensional gauge theory is too difficult to handle.

In string theory, which has Hagedorn behavior at zero coupling this supposed relation has even been used to model what might happen above the Hagedorn temperature [3], which for closed string theories is difficult to interpret as a true limiting temperature, since the energy density never diverges [4]. To actually demonstrate that the Hagedorn and the deconfinement temperatures are identical, and that the thermodynamics below this critical temperature can equally well be described by an exponential density of states or some mechanism binding otherwise free colour charges, requires large computational power, or a very simple model.

$\mathcal{N} = 4$ Super-Yang-Mills at zero coupling on a three-sphere is simple enough for explicit calculations, but is also smoothly related to the interacting theory. The dimensionless coupling is an actual parameter of the theory because of conformal invariance. Although we perform an explicit calculation for this theory most of the calculations of this paper are phrased in terms of a much more general class of toy models. The $\mathcal{N} = 4$ is however very interesting because the AdS/CFT correspondence [5, 6, 7, 8] can be used to discuss gravitational and string theory interpretations of all SYM results, cf. sect. (7). For strong coupling gauge theory this correspondence has been used by Witten [7, 9] in the other direction, for extracting information about gauge theory deconfinement. Furthermore, high and low temperature limits at weak coupling, but not intermediate temperatures, are discussed in [9]. Corrections to the strong coupling results have been studied in [10, 11, 12], with results that do not match well to ours. A possible source of the discrepancy is that these authors consider string corrections to the metric, but not the effects of string spectra on the thermodynamics. On the contrary, the high temperature expansion in [13] seems to be consistent with the similarity between strong and weak coupling results which we find. The implications of the AdS/CFT correspondence for the relation between the Hagedorn transition and deconfinement has been investigated in a different way in [14].

We always study the large N limit of gauge theories with constituent states in the adjoint representation of the gauge group. The other essential ingredients are confinement, and a discrete energy spectrum with a positive energy ground state for

the constituents. There are no interactions among the colour charges in the model except for a singlet constraint from confinement. This makes the model exactly calculable. Hopefully the results are stable to the introduction of small interaction terms. The models we are considering are slight generalizations of singlet ideal gases in flat space [15], and two-dimensional lattice gauge theories with a Wilson action in the adjoint representation. The crucial difference that makes us able to directly identify the Hagedorn behaviour is that we calculate the fluctuations around the leading large N result below the phase transition.

An important feature in our model is that deconfinement never has to take place on the level of states, i.e. no non-singlets have to be taken into account above the transition. In string language there is no "string breaking" above the Hagedorn transition. Instead, it seems that a finite N leads to a density of states that is only approximately exponential, so that a sufficiently high temperature discloses a number of constituent degrees of freedom which is not infinite but $N^2 - 1$. There appears to be a string exclusion principle [16] at work. Perhaps this is a more general property of "deconfinement".

The paper is organized as follows. In section (2) the problem of counting singlet states is formulated and solved for general spectra of constituents. In section (3) it is demonstrated that Hagedorn behaviour results from these state spaces, and the grand canonical partition function close to the Hagedorn temperature is estimated. Partition sums for our prime example, $\mathcal{N} = 4$ SYM, are calculated in section (4). The next section (5) deals with the problem of directly projecting the full grand canonical partition function to the physical singlet states in the large N limit. The result is used to calculate the phase transition temperature and establish the connection to Hagedorn behaviour. Then in section (6) there is a short discussion of the role of unbroken symmetry under the center of the gauge group Z_N . In section (7) connections are made with gravitational physics and we argue that the similarity with between the present weak coupling results and Witten's results at strong coupling [7, 9] indicates that one could also discuss black holes in the AdS dual of the weak coupling theory. It is also argued that AdS black hole thermodynamics is related to Hagedorn thermodynamics. Finally in section (8) we state our conclusions.

2 Counting single trace operators

Let us assume that we have constituent degrees of freedom with discrete energy spectra. Temporarily disregarding the colour degrees of freedom and the singlet condition we can write down partition sums ζ_B and ζ_F for constituent bosons and fermions, in terms of the energies $E_{B,n}$ and $E_{F,n}$ for all their states,

$$\zeta_B(x) = \sum_{n=1} x^{E_{B,n}}, \quad (1)$$

$$\zeta_F(x) = \sum_{n=1} x^{E_{F,n}}, \quad (2)$$

where we have defined

$$x \equiv e^{-\beta} \equiv e^{-1/T}. \quad (3)$$

If we then assume that these fields are in the adjoint representation of $SU(N)$, they can be represented by traceless hermitean matrices, and all gauge invariant states can be written as products of traces of hermitean matrices, each labelled by one of the discrete constituent states. The single-trace states

$$|i_1, i_2, \dots, i_n\rangle = \text{Tr}(\phi_{i_1} \phi_{i_2} \dots \phi_{i_n}) |i_1\rangle |i_2\rangle \dots |i_n\rangle \quad (4)$$

play the role of bound states, with the singlet condition being the only interaction we consider. They are bosons or fermions depending on whether they contain an even or odd number of constituent fermions. Because of the trace, cyclical permutations of the constituent-state labels give rise to the same bound state. The cyclicity may be taken into account by thinking of bound states as "necklaces" of constituent "beads". For finite groups there are in general many relations between the traces. In case of a single matrix, $\text{Tr}(A - \lambda_1)(A - \lambda_2) \dots (A - \lambda_{N-1}) = 0$ gives such a relation since sums of powers of the eigenvalues λ_i are traces of powers of A . In the large N limit of $SU(N)$ only two basic kinds of relations survive: $\text{Tr} A = 0$ for any constituent state A due to tracelessness, and $\text{Tr} \psi^2 = 0$ from the Pauli principle for fermionic constituents ψ . The reason that the Pauli principle is effective for squared constituents but not in other products involving fermions is that only the trace of the square forces a multiplication of identical fermion Lie algebra components. In other cases the fermions can be in different Lie algebra directions.

The bound state partition sum may now be obtained by counting necklaces of constituent states, weighting each energy level with the corresponding Boltzmann weight x , and finally subtracting the constituent partition sum and the sum for two identical fermion constituents. Fortunately, the necklace problem is a special case of a class of combinatorial problems solved by Pólya's theorem [17]. The number of bound states with n constituents can be found from the partition sum

$$\frac{1}{n} \sum_{k|n} \varphi(k) (z_B(x^k) + z_F(x^k))^{n/k}, \quad n > 2, \quad (5)$$

over divisors of n , where φ is called the Euler totient function. $\varphi(n)$ is the number of positive integers less than n which are relatively prime to n , with $\varphi(1) = 1$ by definition. Summing over numbers of constituents and taking due care of the special constraints for few constituents one obtains

$$z(x) = -(\zeta_B(x) + \zeta_F(x)) - \zeta_F(x^2) + \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k|n} \varphi(k) (\zeta_B(x^k) + \zeta_F(x^k))^{n/k} \quad (6)$$

$$= -\zeta_B(x) - \zeta_F(x) - \zeta_F(x^2) - \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log [1 - \zeta_B(x^k) - \zeta_F(x^k)]. \quad (7)$$

3 The Hagedorn transition

One may see quite easily that the number of states grows exponentially with energy if there are at least two constituent states. For n the number of factors inside a trace, the cyclicity relation becomes less and less important for large n and can at most give a factor $1/n$. Neglecting this factor the number of states grows at least as 2^n , and the energy is at most nE_2 , where E_2 is the second lowest energy. Therefore the density of states grows at least as $2^{E/E_2}$.

More precisely, the partition sums (7) have lowest temperature singularities at

$$1 = \zeta(x_H) \equiv \zeta_B(x_H) + \zeta_F(x_H). \quad (8)$$

Close to this point we may approximate

$$z(x) \approx -\log[1 - \zeta(x)] \approx \log[(x_H - x)\zeta'(x_H)], \quad (9)$$

and obtain the contribution to the asymptotic level degeneracy

$$d(n) = \frac{i}{2\pi} \oint \frac{dx}{x^{n+1}} \log[1 - \zeta(x)] \approx \frac{i}{2\pi} \oint \frac{dx}{x^{n+1}} \log[(x_H - x)\zeta'(x_H)] \quad (10)$$

$$\approx -\frac{x_H^{-n}}{2\pi i} \oint \frac{dy}{y^{n+1}} \log[1 - y] \quad (11)$$

in case the spectrum is integrally spaced. The asymptotic density of states should have the same Hagedorn behaviour at $x = x_H$ irrespective of this simplifying assumption.

Because there are so many different states that are easily excited close to the Hagedorn temperature, the effects of identical particles in Bose or Fermi statistics vanishes for the singular behaviour at this temperature. Using Boltzmann statistics we then find the the grand canonical partition function

$$Z(x) \approx \sum_{n=1}^{\infty} \frac{1}{n!} z(x)^n = e^{z(x)} \approx \frac{1}{(x_H - x)\zeta'(x_H)} \quad (12)$$

close to T_H .

4 $\mathcal{N} = 4$ Super Yang-Mills partition sum on S^3

Due to conformal invariance it is simple to find the spectrum of $\mathcal{N} = 4$ SYM on a three-sphere at zero coupling. There are ambiguities in how a field theory should be coupled to a background curvature, although the leading long distance behaviour is fixed by the equivalence principle. For SYM, the ambiguity may be fixed by asking that scalars are conformally coupled.

Then one may take over results from flat spacetime to $S^3 \times \mathbf{R}$ by a conformal mapping [9]. The generator of scale transformations in the flat Euclidean theory is

mapped to the Hamiltonian on the sphere. Because these operators are also related by an isomorphism of the conformal group they have identical spectra. We can conclude that energies of states on the sphere (of unit radius) are given by scaling dimensions of operators in flat spacetime. (In two dimensions this is the mapping between the plane and the cylinder.)

On the sphere, all states have to be gauge singlets because of the Gauss law constraint and the lack of a boundary. We can then use the general procedure above to find the bound state partition sum and the grand canonical partition function close to the Hagedorn temperature. We only need the constituent Bose and Fermi partition sums. The constituent fields, scalars, fermions and gluons, their respective scaling dimensions and their partition sums are given in

<i>Field</i>	<i>Dimension</i>	$(1-x)^4 \zeta(x)$
Φ	1	$6x - 6x^3$
λ	$\frac{3}{2}$	$8x^{3/2} - 8x^{5/2}$
$\bar{\lambda}$	$\frac{3}{2}$	$8x^{3/2} - 8x^{5/2}$
F	2	$6x^2 - 8x^3 + 6x^4$

(13)

where the higher terms in the sums arise from counting possible derivative operators (conformal descendants). Terms corresponding to equations of motion and their derivatives have been subtracted, since they give rise to trivial correlation functions². For gluons we have also taken into account Bianchi identities and the antisymmetry of the field strength. We are now ready to write down the partition sums for free $\mathcal{N} = 4$ SYM at the limit $N = \infty$:

$$\zeta_B^{SYM}(x) = \frac{6x + 6x^2 - 14x^3}{(1-x)^4} \quad (14)$$

$$\zeta_F^{SYM}(x) = \frac{16x^{3/2} - 16x^{5/2}}{(1-x)^4} \quad (15)$$

$$z_B^{SYM}(x) = -\zeta_B^{SYM}(x) - \zeta_F^{SYM}(x^2) \quad (16)$$

$$- \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log \left[1 - 2\zeta_B^{SYM}(x^k) + \zeta_B^{SYM}(x^k)^2 - \zeta_F^{SYM}(x^k)^2 \right] \quad (17)$$

$$z_F^{SYM}(x) = -\zeta_F^{SYM}(x) - \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log \left[\frac{1 - \zeta_B^{SYM}(x^k) - \zeta_F^{SYM}(x^k)}{1 - \zeta_B^{SYM}(x^k) + \zeta_F^{SYM}(x^k)} \right]. \quad (18)$$

The Bose and Fermi partition sums are obtained by keeping track of whether the states contain an even or odd number of constituent fermions. The Hagedorn temperature is then given by the lowest temperature singularity, which can be found by solving eq. (8). We find that $x_H \approx 0.072$. Furthermore, the grand canonical partition function approaches (12) close to T_H .

²One may check, most simply for a conformal scalar on S^2 , that this subtraction gives the correct counting of states.

All these calculations were performed at zero coupling, but one may hope that our general toy models also capture some of the essential ingredients of the interacting case, perhaps by renormalization of the constituent spectra.

5 The ideal gas with singlet condition

The scale of T_H is set by $1/R$, the inverse radius of the sphere, which is the only fundamental energy scale in the problem. Thus $T_H \rightarrow 0$ in the flat space limit, and we seem to get a divergent free energy at any temperature! The argument for considering only gauge singlet states breaks down in a non-compact space, but on the other hand a singlet condition ought to make little difference in infinite volume. And we do not really want to consider non-singlet states in a non-Abelian gauge theory. At best they are going to be extremely sensitive to turning on the interactions.

Instead, the problem is due to setting $N = \infty$. The free energy at zero coupling in flat space is well known, and is proportional to $N^2 - 1$. To go above the Hagedorn temperature we have to take the large N limit in a way that allows us to extract divergent factors of N^2 . We should also compute quantities like the free energy, which scale simply with N , rather than the bound state partition sum, which certainly changes with N due to trace identities, but in a much less regular fashion.

We write down the grand canonical partition function with a singlet constraint as

$$\int_{SU(N)} dg \exp \left(\sum_{n=1}^{\infty} \frac{\zeta_B(x^n)}{n} \chi(g^n) - \sum_{n=1}^{\infty} (-1)^n \frac{\zeta_F(x^n)}{n} \chi(g^n) \right) \quad (19)$$

where g are elements of $SU(N)$ and $\chi(g)$ is the character of the adjoint representation. This equation can be derived in a general way by a coherent state technique [15]. Alternatively, one may check that the expression represents the proper combinatorics, by noting that singlets can be identified by acting with $SU(N)$ transformations on the constituent states. It is enough to keep track of the eigenvalues $R_i(g)$ of the representation matrices $R(g)$. The rotated partition sum for bosons

$$\prod_k \prod_{i=1}^{N-1} (1 + x^{E_k} R_i(g) + x^{2E_k} R_i(g)^2 + \dots) \quad (20)$$

$$= \prod_k \text{Det}(1 - x^{E_k} R(g))^{-1} = \exp \left(\sum_k \text{Tr} \left(\sum_{n=1}^{\infty} \frac{x^{nE_k}}{n} R(g^n) \right) \right) \quad (21)$$

can be projected to singlet states by integration over the group, using the orthogonality properties of group characters $\chi(g)$. Taking also the fermion contribution into account yields eq. (19).

In terms of the eigenvalues $\exp(i\alpha_i)$ of g the adjoint character

$$\chi(g) = -1 + \left(\sum_{m=1}^N e^{i\alpha_m} \right) \left(\sum_{n=1}^N e^{-i\alpha_n} \right) = N - 1 + 2 \sum_{m < n} \cos(\alpha_m - \alpha_n). \quad (22)$$

The fact that the integrand only depends on the eigenvalues simplifies the group integral to an integral over these eigenvalues

$$\frac{1}{N!} \int \prod_{i=1}^N \frac{d\alpha_i}{2\pi} \prod_{i \neq j} \left| 2 \sin \left(\frac{\alpha_i - \alpha_j}{2} \right) \right| \left(\sum_{k=1}^N \delta \left(-\frac{2\pi k}{N} + \sum_i \alpha_i \right) \right) \quad (23)$$

$$\times \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} (\zeta_B(x^n) - (-1)^n \zeta_F(x^n)) \left(-1 + \sum_{i,j} \cos(n(\alpha_i - \alpha_j)) \right) \right). \quad (24)$$

The sum inside the δ function encodes that we are dealing with $SU(N)$ rather than $SU(N)/Z_N$. By standard large N techniques the discrete set of eigenvalues may be replaced by an eigenvalue density ρ such that

$$\rho(x) = \frac{1}{N} \frac{dn}{d\alpha_n}, \quad (25)$$

$$1 = \int \rho(\alpha) d\alpha. \quad (26)$$

Since different orderings of the eigenvalues are related by Weyl transformations it is enough to consider a single ordering, which may be chosen increasing, thus making ρ positive. In the large N limit the δ function in (24) becomes irrelevant unless we are dealing with sharply peaked eigenvalue distributions which occur for very large temperatures. For the time being we can neglect this factor. Then we find a partition function

$$Z(x) \propto \int D\rho e^{-S[\rho]}, \quad (27)$$

where

$$S[\rho] = -N^2 \int d\alpha \int d\beta \rho(\alpha) K(\alpha, \beta; x) \rho(\beta) + \sum_{n=1}^{\infty} \frac{1}{n} (\zeta_B(x^n) - (-1)^n \zeta_F(x^n)), \quad (28)$$

$$K_x(\alpha, \beta) \equiv \log \left| 2 \sin \left(\frac{\alpha - \beta}{2} \right) \right| + \sum_{n=1}^{\infty} \frac{1}{n} (\zeta_B(x^n) - (-1)^n \zeta_F(x^n)) \cos(n(\alpha - \beta)) \quad (29)$$

can be regarded as an effective action for the eigenvalue density. The stationarity condition in a steepest decent estimate for Z then gives the integral equation

$$0 = \text{p.v.} \int d\beta \left\{ \frac{1}{2} \cot \left(\frac{\alpha - \beta}{2} \right) - \sum_{n=1}^{\infty} (\zeta_B(x^n) - (-1)^n \zeta_F(x^n)) \sin(n(\alpha - \beta)) \right\} \quad (30)$$

for ρ . (One does not get the integral of this equation since ρ satisfies an additional integral constraint.) Since the equation is translationally invariant in α -space $\rho \equiv 1/2\pi$ is always a solution, but there may be others with lower free energy, which thus would appear to break the translational symmetry spontaneously. As we shall see below, this is almost what happens for high enough temperature, but not quite.

To identify the transition point where non-trivial solutions appear, it is useful to view the effective action as a quadratic expression in ρ . We can regard

$$K_x : \rho(\alpha) \rightarrow K_x \rho(\alpha) \equiv - \int d\beta K_x(\alpha, \beta) \rho(\beta) \quad (31)$$

as a symmetric operator. Then, modulo the additional conditions on ρ , the search for the minimum of the free energy becomes the search for the lowest eigenvalue of K_x . There will be a transition point at a value of the temperature where the lowest K_x -eigenvalue becomes degenerate. Such a point is easy to find by Fourier expanding the K_x -eigenvalue equation. The K_x -eigenfunctions are found to be trigonometric functions:

$$K_x e^{in\alpha} = \frac{\pi}{n} (1 - z_B(x^n) + (-1)^n z_F(x^n)) e^{in\alpha}, \quad n \neq 0 \quad (32)$$

$$K_x 1 = 0 \quad (33)$$

We find a transition point at an $x = x_D = x_H$ satisfying equation (8) determining the Hagedorn temperature.

The K_x -eigenfunctions which become degenerate with the constant solution indicate the nature of the transition. We have to require $\rho \geq 0$, as ρ is an eigenvalue density, which should also integrate to 1. Therefore, the new K_x -eigenfunctions are not acceptable as such. Instead, the linear combinations

$$\rho(\alpha) = \frac{1}{2\pi} (1 + a_1 \cos(\alpha) + a_2 \sin(\alpha)), \quad a_1^2 + a_2^2 \leq 1 \quad (34)$$

are the minimizing solutions at the transition point. At the boundary of the solution set we find the solutions $\rho(\alpha) = \sin^2(\alpha + \delta)/\pi$, which are closest to the new K_x -eigenfunctions. Apparently, at the transition it becomes energetically possible for the density to have a zero. On the other side of the transition the density can still not become negative, but the eigenvalue distribution can have "gaps", where the density vanishes. To solve this case, which involves new boundary conditions, a more direct approach to the integral equation (eq. 30) is needed.

This kind of boundary problem is commonplace in large N calculations [18, 19]. A somewhat unusual feature of equation (30) is that the two-eigenvalue interaction is non-trivial, whereas a non-trivial single-eigenvalue potential is more familiar. Fortunately, there is still a mathematical theory for this kind of problem [20]. Essentially, the mathematical method consists in multiplying the integral equation (30) with an operator that is inverse to the first, singular term, in order to get a regular equation of Fredholm type. This equation either lacks solutions or has a finite-dimensional solution space, which may be searched for solutions to the original equation.

The higher, $n > 1$, Fourier components of the non-singular term in the integral equation (30) are exponentially small relative to the $n = 1$ term at finite temperatures, and go as inverse powers of n for asymptotically high temperatures, at least when the constituent density of states is asymptotically constant or increasing³. It is then reasonable to neglect the higher terms and study the approximate equations obtained by replacing the kernel $K_x(\alpha, \beta)$ with

$$K_x^1(\alpha, \beta) \equiv \log \left| 2 \sin \left(\frac{\alpha - \beta}{2} \right) \right| + (z_B(x) + z_F(x)) \cos(\alpha - \beta) \quad (35)$$

³This is the case for free field theory in one or more spatial dimensions.

Precisely this mathematical problem is solved in [15]. We find the same solution, but also note a misprint in eq. (19) of that reference. Instead, the free energy $F = -\log Z$ above the transition reads

$$F_{>} \approx -\frac{N^2}{2} \left(\zeta(x) - 1 + \sqrt{\zeta(x)^2 - \zeta(x)} - \log \left(\zeta(x) + \sqrt{\zeta(x)^2 - \zeta(x)} \right) \right) + \zeta(x) \quad (36)$$

We also give the solutions for the eigenvalue densities

$$\rho_{\delta}(\alpha) = \rho_0(\alpha - \delta) \quad (37)$$

$$\rho_0(\alpha) = \begin{cases} \frac{\cos(\frac{1}{2}\alpha)}{\pi \sin^2(\frac{1}{2}\alpha_c)} \sqrt{\sin^2(\frac{1}{2}\alpha_c) - \sin^2(\frac{1}{2}\alpha)}, & |\alpha| < \alpha_c \\ 0, & |\alpha| > \alpha_c \end{cases} \quad (38)$$

$$\sin^2(\frac{1}{2}\alpha_c) = 1 - \sqrt{1 - \frac{1}{\zeta(x)}} \quad (39)$$

At the transition, these solutions to an approximate problem agrees with the extreme cases of the exact solutions (34) to the exact problem. We do however only expect qualitative and approximate agreement above the transition. If we calculate the effective action (29) below the transition we can also estimate the free energy

$$F_{<} = \zeta(x) \quad (40)$$

which together with eqs. (36) signifies a first order large N transition at the Hagedorn temperature.

The N^2 dependence of the high temperature phase agrees with expectations of deconfinement, and the "deconfinement" transition takes place at the Hagedorn temperature, but eqs. (36,40) still do not represent a satisfactory state of affairs. The large N approximation of eq. (24) does not give Hagedorn behaviour.

The resolution to this puzzle is that the Hagedorn and the deconfinement transitions can be identified, but that one has to go to next order in the large N expansion to see the Hagedorn behaviour. Actually, although eq. (40) results from the leading estimate of eq. (24) they are of subleading order in $1/N$ and do not give the full N^0 contribution to thermodynamics. Fluctuations in the eigenvalue density give contributions of the same order and should also be taken into account. Below T_H this is remarkably easy (and above we do not have to do the calculation, if the goal is just to find Hagedorn thermodynamics).

The integral over eigenvalue-density fluctuations can only diverge at T_H because of the contribution from the integral over fluctuations in the eigen-directions corresponding to the eigenvalues that become degenerate with the low temperature solution at T_H . The integral over these fluctuations given in eq. (34) yields the correction factor

$$\sim \frac{1}{(1 - \zeta(x))} \left(1 - e^{-N^2(1 - \zeta(x))} \right), \quad \zeta(x) < 1 \quad (41)$$

to the partition function, which diverges at the Hagedorn transition if the large N limit is taken before the approach to T_H . Otherwise, the approach to the "deconfined" phase is fast but smooth. This is just how the integrals tell us that an infinite number of degrees of freedom are needed for a true phase transition, at finite volume an infinite N is needed. With this reservation complete agreement is found with eq. (12), which of course was derived assuming an infinite N .

6 Z_N symmetry

We have only associated the two phases of our model with confined and deconfined phases on the basis of the N dependence of the free energy. This is reasonable, but it should be compared to other criteria. A Polyakov loop, i.e. a Wilson loop around the compact imaginary time direction, is an order parameter for deconfinement. Z_N symmetry (the centre of the $SU(N)$ symmetry) acts by multiplication by a root of unity on the Polyakov loop. Thus the loop has a vanishing expectation value in the confined phase, which enjoys unbroken Z_N symmetry. In the deconfined phase it gets a non-zero vacuum expectation value, due to spontaneous breaking of Z_N symmetry. Using the AdS/CFT correspondence the Polyakov loop has been studied at strong coupling in [9, 21]. Our simple formalism is not well suited to a calculation of the loop, but there are alternatives.

In our fluctuation calculation unbroken Z_N symmetry is essential to get agreement with the Hagedorn calculation in the confined phase. The symmetry appears in the integration over α -translations, which results in a two-dimensional integral over fluctuations, giving the correct degree of divergence. On the high temperature side we have no similar check on the calculation. The best we can do is to judge from the partition sum itself how Z_N symmetry is represented. The eigenvalue densities (39) clearly break α -translation symmetry, so the issue is if the partition sum should be thought of as a sum over all possible translated minima of the effective action and fluctuations around them, or if a single minimum is selected. The simple answer is that there is no spontaneous breaking of symmetry in finite volume, even if N is large. A more satisfactory answer is that the high temperature limit, which in the conformal $\mathcal{N} = 4$ SYM theory is the same as the infinite volume limit, eventually leads to the support $[2\pi n/N + \alpha_c, 2\pi n/N - \alpha_c]$ of the equilibrium solutions for the eigenvalue densities (39) narrowing so much that there is no overlap between densities related by Z_N transformations (different n from the δ function in (24)). Even if the finite volume sum always is Z_N symmetric, this behaviour allows for a decoupling of fluctuations around different Z_N -related minima, which has to happen when Z_N is broken spontaneously. Since the high temperature phase is thus smoothly connected to a phase with spontaneously broken Z_N , it makes sense to describe it as deconfined.

7 AdS/CFT motivated speculations

There is as yet no good test of the AdS/CFT correspondence [5, 6, 7, 8] at weak coupling, so we cannot make strong predictions about gravitational effects. We can however try to use the AdS/CFT dictionary, first to compare with strong coupling results obtained via the correspondence, and second to find out what it tells us about a dual gravitational theory, assuming that the correspondence holds also at small 't Hooft coupling g^2N , which means small string tension relative to the AdS curvature.

At strong coupling Witten [7, 9] has used the AdS/CFT correspondence to discuss a large N phase transition at finite temperature. At leading order in large N he finds a high temperature phase with free energy proportional to N^2 and a low temperature phase where the (temperature dependent part of) the N^2 term in the free energy vanishes. This precisely the behaviour we observe at weak coupling. It seems that thermodynamics on S^3 can have a smooth dependence on the coupling. In the flat case there is a debate about whether there is a phase transition in the coupling [22] or not [23, 24, 25, 26, 27], and although most explicit calculations at extreme temperatures indicate a smooth behavior, the issue does not seem to be settled.

Witten's result is obtained by associating the gauge theory behaviour with a gravitational phase transition between an AdS heat bath and an AdS black hole, found by Hawking and Page [28]. It is of course tempting to identify the Yang-Mills thermodynamics we have found, with a similar black hole transition on the gravitational side, but now in a theory of tensionless strings. Interpreted in gravitational language the N^2 dependence of the free energy translates to the ordinary $1/G_N$ dependence of a black hole free energy. The entropy would give an effective measure of the horizon area of this exotic black hole.

The N^0 corrections correspond to Hagedorn behaviour of tensionless strings⁴, and beyond the Hagedorn transition we have a black hole equilibrium state. Such a phase diagram have been proposed for non-zero string tension [29], by arguments based on a black hole/string correspondence principle [30]. Our explicit calculation is done at vanishing gauge coupling, but since N^0 corrections in the low temperature phase ought to be present independently of the coupling, this mechanism could give a general confirmation of the relation between Hagedorn spectra and black holes. It might seem strange that a string Hagedorn transition could occur at a scale governed by the AdS curvature rather than the string tension, but most of the important part of the Hagedorn spectrum consists of large strings, which should be more sensitive to curvature than to the tension. A complete picture might have to await the quantization of strings in $\text{AdS}_5 \times S^5$ with background Ramond-Ramond fields.

⁴Their spectra are governed the curvature scale rather than the string scale.

8 Conclusions

We have noted a general mechanism in large N theories, which can be important whenever the N^2 contribution to the free energy vanishes on one side of a phase transition. On this side, fluctuation can give an N^0 contribution which diverges at the transition. We have done this calculation explicitly for free $\mathcal{N} = 4$ Super Yang-Mills theory on a three-sphere, and for quite a general class of toy models. The large N phase transition we find is a kind of deconfinement transition, in which Hagedorn behaviour in the confined phase is identified as a low-temperature precursor of deconfinement.

Assuming the AdS/CFT correspondence is valid also for small or vanishing 't Hooft coupling, the transition has the likes of a Hawking-Page transition to AdS black holes, although in a string theory with vanishing tension. The N^0 correction we have calculated should then correspond to a Hagedorn contribution from an ideal gas of these strings.

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